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Chains of dog-ears for completely positive matrices

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Abstract

A necessary and sufficient condition to determine the complete positivity of a matrix with a particular graph, in dependence of complete positivity of smaller matrices, is given. Under some singularity assumptions, this condition furnishes a characterization for completely positive matrices with a "non-crossing cycle" as associated graph. In particular the characterization holds for singular pentadiagonal matrices. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

A real symmetric $n \times n$ matrix A is said to be *completely positive* if there exists a non-negative $k \times n$ matrix V such that $A = V^{\mathrm{T}}V$. The completely positive matrices have been studied beginning from the 1960s (see [4,5,7]) as the dual cone of the copositive matrices.

Actually the main problem is to decide if a given matrix is or is not completely positive. A necessary condition for a matrix to be completely positive is to be doubly non-negative, namely positive semidefinite and entrywise non-negative. The

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same condition is also sufficient only if $n \leqslant 4$ (see [6]), or rank $A \leqslant 2$. For higher orders and ranks there are many examples of doubly non-negative matrices which fail to be completely positive. Since for reducible matrices the complete positivity is equivalent to the complete positivity of each irreducible diagonal block, the problem reduces to study irreducible matrices. Moreover we recall that a cogredience, that is a simultaneous permutation of corresponding rows and columns, does not alter complete positivity.

Interesting results were obtained in [2,3,8] by considering the associated graph of A, $G(A)$, that is the non-directed graph with n vertices such that the edge (i, j) belongs to $G(A)$ if and only if $a_{ij} \neq 0$. In particular if $G(A)$ does not contain odd cycles of length greater than 3, the double non-negativity is again a sufficient condition for the complete positivity. These kinds of graphs are called *completely positive graphs*. Conversely, if G is a graph containing an odd cycle of length greater than 3, there always exists a doubly non-negative matrix A, with $G(A) = G$, which fails to be completely positive.

So the problem arises when we are going to move away from the class of the completely positive graphs. In this case two different approaches are presented, respectively, in [1] and [3]. In [1] the idea is to bring some completely positive graphs and join them together by identifying an edge. The resulting graph is called *book-graph*. For a matrix with a book-graph the complete positivity depends on some simple inequalities. In [3], on the contrary, the idea is to move from a graph to a smaller and simpler one, by deletion of one or more vertices.

In our paper we carry on this last idea, finding a more general way to delete vertices and edges in a graph. As we will see the technique will work successfully for singular matrices. In Section 2, after some notations, we just present the *deletion of a leaf* of Berman–Hershkowitz [3], and a technical lemma very useful in the following sections. In Section 3 we consider graphs with a dog-ear, namely a vertex directly connected with exactly two other vertices. We also introduce the notion of diminishability for a matrix, that is a particular condition on the rank. The complete positivity of a diminishable matrix is shown to be equivalent to the complete positivity of a smaller matrix whose graph is obtained from the original graph by deleting the dog-ear. In Section 4, in order to apply inductively the deletion of a dog-ear, we define a new kind of graph, the *non-crossing cycle* which is basically a sequence of triangles connected to each other by an edge. A particular non-crossing cycle is the graph of a pentadiagonal matrix. So for a singular pentadiagonal matrix, all of whose second-super diagonal entries are non-zero, the complete positivity is equivalent to the non-negativity of a sequence of positive semidefinite matrices. Finally, in Section 5, we try to remove the diminishability assumption. However, as we will see, the deletion of a dog-ear does not work any longer. The remaining part of the section is devoted to investigate the reason for this failure. This allows us to present a possible way to go, in order to extend the results to the non-singular case.

2. Deletion of a leaf

We just quote some notations already introduced in [3].

Notation 2.1. Let n be a positive integer. We denote:

$$
\begin{aligned}
\langle n\rangle &= \text{the set } \{1, 2, \ldots, n\};\\
E_{ij} &= \text{the } n\times n \text{ matrix in which the unique non-zero entry is}\\
&\quad\ \text{the one in the } (i, j) \text{ position, whose value is } 1;\\
\mathscr{DN}^{(n)} &= \text{the set of the } n\times n \text{ doubly non-negative matrices};\\
\mathscr{CP}^{(n)} &= \text{the set of the } n\times n \text{ completely positive matrices.}
\end{aligned}
$$

Notation 2.2. Let A be an $n \times n$ matrix and let $\alpha, \beta \subseteq \langle n\rangle,\ \alpha, \beta \neq \emptyset$. We denote:

$$
\begin{aligned}
A[\alpha \mid \beta] &= \text{the submatrix of } A \text{ obtained by taking only the rows}\\
&\quad\ \text{and columns indexed, respectively, by } \alpha \text{ and } \beta;\\
A[\alpha] &= A[\alpha \mid \alpha];\\
A(\alpha \mid \beta) &= A[\langle n\rangle\backslash\alpha \mid \langle n\rangle\backslash\beta];\\
A(\alpha) &= A(\alpha \mid \alpha);\\
A(i) &= A(\{i\}), \text{ where } i \in \langle n\rangle;\\
R(A) &= \text{the column space of } A;\\
A^{+} &= \text{the Moore–Penrose pseudoinverse of } A;\\
\mathbf{O}_{m,n} &= \text{the } m\times n \text{ matrix all of whose entries are } 0.
\end{aligned}
$$

When the size will be obvious by the context, we will write $\mathbf{O}$ instead of $\mathbf{O}_{m,n}$.

Notation 2.3. Let A be an arbitrary matrix and $\underline{v}$ a vector. We denote:

$$
\begin{aligned}
(A)_{ij} &= \text{the entry of } A \text{ in the } (i, j) \text{ position};\\
(A)_{*j} &= \text{the } j\text{th column of } A;\\
(\underline{v})_i &= \text{the } i\text{th component of } \underline{v}.
\end{aligned}
$$

Notation 2.4. Let A and B be two $n \times n$ positive semidefinite matrices. We write $A \geqslant B$ or $A - B \geqslant 0$ if $A - B$ is positive semidefinite.

Definition 2.5. Given two vertices i, j in a connected graph G, we say that i is (directly) *connected* to j if $(i, j) \in G$.

Definition 2.6. A vertex i of a graph G is said to be a *leaf* if there is a unique vertex $j \neq i$ connected to i.

The complete positivity of a matrix A whose graph $G(A)$ has a leaf is always equivalent to the complete positivity of a smaller matrix whose graph is $G(A)$ deprived of the leaf itself. This is the statement of the following.

Theorem 2.7 [3, Theorem 4.1]. *Let A be a non-negative symmetric matrix, and suppose that $G(A)$ has a leaf i connected to the vertex j. Then A is completely positive if and only if the matrix*

$$A^{(1)} = \left[A - \frac{a_{ij}^2}{a_{ii}} E_{jj} \right](i) \tag{1}$$

is completely positive.

The following lemma in practice claims that the addition of a 2×2 positive semidefinite submatrix does not remove the complete positivity. Even if it looks as a predictable result, it is the real basis of all the following results.

Lemma 2.8. *Let $A \in \mathscr{CP}^{(n)}$. Let $p, q \in \langle n \rangle$ and H be an $n \times n$ positive semidefinite matrix such that $h_{ij} = 0$ if $\{i, j\} \not\subseteq \{p, q\}$. If $A + H$ is non-negative, then $A + H$ is completely positive.*

Proof. Without loss of generality we can assume $p = 1$ and $q = 2$. So

$$H = \begin{bmatrix} h_{11} & h_{12} & \underline{0}^{\mathrm{T}} \\ h_{12} & h_{22} & \underline{0}^{\mathrm{T}} \\ \underline{0} & \underline{0} & \mathbf{O} \end{bmatrix}.$$

If H is non-negative, the proof is trivial. So let us suppose $h_{12} < 0$, while clearly $a_{12} + h_{12} \geqslant 0$. We can also assume rank $H = 1$. In fact if rank $H = 2$, we can write $H = H' + H''$, where

$$H' = \begin{bmatrix} h_{11} & h_{12} & \underline{0}^{\mathrm{T}} \\ h_{12} & \frac{h_{12}^2}{h_{11}} & \underline{0}^{\mathrm{T}} \\ \underline{0} & \underline{0} & \mathbf{O} \end{bmatrix}, \qquad H'' = \begin{bmatrix} 0 & 0 & \underline{0}^{\mathrm{T}} \\ 0 & h_{22} - \frac{h_{12}^2}{h_{11}} & \underline{0}^{\mathrm{T}} \\ \underline{0} & \underline{0} & \mathbf{O} \end{bmatrix}.$$

Now rank $H' = 1$ and $A + H'$ is still non-negative. If we prove $A + H'$ to be completely positive, by $A + H = (A + H') + H''$ we conclude. So let us assume rank $H = 1$. Since $A \in \mathscr{CP}^{(n)}$, we can write $A = \sum_1^t \underline{c}_i \underline{c}_i^{\mathrm{T}}$ for some $t > 0$, $\underline{c}_i \geqslant 0$. Furthermore $H = \underline{d}\,\underline{d}^{\mathrm{T}}$ for some $\underline{d} \in \mathbb{R}^n$. For any $1 \leqslant i \leqslant t$ we define

$$\underline{d}_i = \sqrt{\frac{(\underline{c}_i)_1(\underline{c}_i)_2}{a_{12}}}\,\underline{d}.$$

Finally for any i we define $A_i = \underline{c}_i \underline{c}_i^{\mathrm{T}} + \underline{d}_i \underline{d}_i^{\mathrm{T}}$. A_i is positive semidefinite. Moreover it is non-negative. The only check is for the (1,2) entry:

$$\begin{aligned} (A_i)_{12} &= (\underline{c}_i)_1(\underline{c}_i)_2 + (\underline{d}_i)_1(\underline{d}_i)_2 \\ &= (\underline{c}_i)_1(\underline{c}_i)_2 + \frac{(\underline{c}_i)_1(\underline{c}_i)_2}{a_{12}}(\underline{d})_1(\underline{d})_2 \\ &= (\underline{c}_i)_1(\underline{c}_i)_2 \left[1 + \frac{h_{12}}{a_{12}} \right] \geqslant 0, \end{aligned}$$

since $a_{12} + h_{12} \geqslant 0$. Now rank $A_i \leqslant 2$ $\forall i$, and so $A_i \in \mathscr{CP}^{(n)}$ $\forall i$. Finally

$$\begin{aligned}\sum_1^t A_i &= \sum_1^t \underline{c}_i\underline{c}_i^{\mathrm{T}} + \underline{d}_i\underline{d}_i^{\mathrm{T}} \\ &= A + \sum_1^t \frac{(\underline{c}_i)_1(\underline{c}_i)_2}{a_{12}}\underline{d}\,\underline{d}^{\mathrm{T}} \\ &= A + \frac{a_{12}}{a_{12}}H \\ &= A + H\end{aligned}$$

and so $A + H$ is the sum of t completely positive matrices. □

3. Deletion of a dog-ear

Definition 3.1. If i, j, k are distinct vertices of a graph G, we say that $(i; j, k)$ is a *dog-ear* for G if i is connected only to j and k. The vertex i is called the *vertex of the dog-ear.*

We can renumber the vertices of G so that the dog-ear is $(1; 2, 3)$. So if A is a symmetric matrix whose graph has a dog-ear, A can be written, after a suitable cogredience, in the form

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \underline{0}^{\mathrm{T}} \\ a_{12} & a_{22} & a_{23} & \underline{r}_2^{\mathrm{T}} \\ a_{13} & a_{23} & a_{33} & \underline{r}_3^{\mathrm{T}} \\ \underline{0} & \underline{r}_2 & \underline{r}_3 & N \end{bmatrix}. \tag{2}$$

Sometimes we will write (2) placing in evidence the first two rows and columns

$$A = \begin{bmatrix} \hat{A} & S^{\mathrm{T}} \\ S & M \end{bmatrix}, \tag{3}$$

where $\hat{A}$ is a 2×2 block.

Definition 3.2. Let A be a doubly non-negative matrix whose graph contains the dog-ear $(1; 2, 3)$. According to (3) A is said to be *diminishable* if rank $A \leqslant$ rank $M + 1$.

In particular if M is non-singular, A is diminishable exactly when it is singular. Moreover we define the matrices

$$A^L = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \underline{0}^{\mathrm{T}} \\ a_{12} & a_{12}^2/a_{11} & a_{12}a_{13}/a_{11} & \underline{0}^{\mathrm{T}} \\ a_{13} & a_{12}a_{13}/a_{11} & a_{13}^2/a_{11} & \underline{0}^{\mathrm{T}} \\ \underline{0} & \underline{0} & \underline{0} & \mathbf{O}_{n-3,n-3} \end{bmatrix}, \tag{4}$$

$$A^D = A - A^L. \tag{5}$$

Finally according to (3) we define

$$A^S = \begin{bmatrix} S^{\mathrm{T}}M^+S & S^{\mathrm{T}} \\ S & M \end{bmatrix}, \tag{6}$$

$$A^R = A - A^S. \tag{7}$$

Note that in A^R only the first 2×2 principal submatrix is non-zero.

Lemma 3.3. *All the matrices A^L, A^D, A^S and A^R are positive semidefinite.*

Proof. Since A is positive semidefinite, we have $A = W^{\mathrm{T}}W$ for some real $k \times n$ matrix W. Now let us write W by columns

$$W = [\underline{w}_1 \;\; \underline{w}_2 \;\; \underline{w}_3 \;\; \cdots \;\; \underline{w}_n],$$

and let us define

$$W_M = [\underline{w}_3 \;\; \ldots \;\; \underline{w}_n],$$

$$W_L = \underline{w}_1\underline{w}_1^+W, \quad W_D = [I_k - \underline{w}_1\underline{w}_1^+]W,$$

$$W_S = W_M W_M^+ W, \quad W_R = [I_k - W_M W_M^+]W.$$

Now an easy check proves $A^X = W_X^{\mathrm{T}}W_X$ for $X = L, D, S, R$. $\square$

Lemma 3.4. *Let $A \in \mathscr{DN}^{(n)}$ be in the form* (2). *If A is diminishable, each positive semidefinite matrix B such that $A^S \leqslant B \leqslant A$ can be written as a convex combination of A^S and A.*

Proof. Since $\operatorname{rank} A^S = \operatorname{rank} M$, the diminishability of A implies $\operatorname{rank} A^R \leqslant 1$. We can write $A^R = A - A^S = (A - B) + (B - A^S)$, thus

$$A - B = \beta A^R, \quad B - A^S = (1-\beta)A^R$$

for some $\beta \in [0, 1]$, so $B = \beta A^S + (1-\beta)A$. $\square$

Definition 3.5. For a matrix A in the form (2), we define the *diminution discriminant* of A as the number

$$\delta(A) = \det\begin{bmatrix} a_{11} & a_{13} \\ a_{12} & a_{23} \end{bmatrix}. \tag{8}$$

Definition 3.6. With regard to (6) and (8), the *diminution coefficient* of A is the number

$$\alpha(A) = \begin{cases} 0 & \text{if } \delta(A) \geqslant 0, \\ \frac{-\delta(A)}{\delta(A^S)-\delta(A)} & \text{if } \delta(A) < 0 \text{ and } \delta(A^S) \geqslant 0, \\ 1 & \text{otherwise.} \end{cases} \tag{9}$$

Clearly, for any A, $0 \leqslant \alpha(A) \leqslant 1$.

Definition 3.7. Let $A \in \mathscr{DN}^{(n)}$ be in the form (2). The matrices

$$A^{(1/2)} = A - \alpha(A)A^R = \alpha(A)A^S + (1-\alpha(A))A, \tag{10}$$

$$A^{(1)} = \left[A^{(1/2)}\right]^D (1) \tag{11}$$

are called, respectively, the *half-diminished* and the *diminished* matrix of A.

As we will see, if A is diminishable, its complete positivity will be equivalent to the complete positivity of $A^{(1/2)}$ and $A^{(1)}$. At the moment let us investigate their double non-negativity.

Proposition 3.8. *The matrix $A^{(1/2)}$ defined in* (10) *is doubly non-negative. Furthermore*

$$\delta(A^{(1/2)}) \begin{cases} \geqslant 0 & \textit{if } \delta(A) \geqslant 0, \\ = 0 & \textit{if } \delta(A) < 0 \textit{ and } \delta(A^S) \geqslant 0, \\ < 0 & \textit{otherwise}. \end{cases} \tag{12}$$

Proof. $A^{(1/2)}$ is positive semidefinite since (10) shows that $A^{(1/2)}$ is a convex combination of A^S and A. In order to prove the non-negativity of $A^{(1/2)}$ it is enough to check its $(1, 2)$ entry. We have the following three cases:

(i) $\delta(A) \geqslant 0$. From (9) and (10), we have $A^{(1/2)} = A$ and so $A^{(1/2)}$ is non-negative and $\delta(A^{(1/2)}) = \delta(A) \geqslant 0$.

(ii) $\delta(A) < 0$ and $\delta(A^S) \geqslant 0$. First let us compute $\delta(A^{(1/2)})$. In order to simplify the notation, the (i, j) entry of A^S will be denoted by a_{ij}^S; moreover we will write α instead of $\alpha(A)$. We have

$$\delta(A^{(1/2)}) = \det\begin{bmatrix} a_{11} - \alpha(a_{11} - a_{11}^S) & a_{13} - \alpha(a_{13} - a_{13}^S) \\ a_{12} - \alpha(a_{12} - a_{12}^S) & a_{23} - \alpha(a_{23} - a_{23}^S) \end{bmatrix}.$$

Since $a_{13}^S = a_{13}$ and $a_{23}^S = a_{23}$, we arrive at

$$\begin{aligned}\delta(A^{(1/2)}) &= a_{11}a_{23} - \alpha a_{23}(a_{11} - a^S_{11}) - a_{12}a_{13} + \alpha a_{13}(a_{12} - a^S_{12})\\ &= \delta(A) - \alpha[a_{11}a_{23} - a_{12}a_{13} - a^S_{11}a_{23} + a^S_{12}a_{13}]\\ &= \delta(A) + \frac{\delta(A)}{\delta(A^S) - \delta(A)}[\delta(A) - \delta(A^S)] = 0.\end{aligned}$$

This implies

$$(A^{(1/2)})_{12} = \frac{(A^{(1/2)})_{11}(A^{(1/2)})_{23}}{(A^{(1/2)})_{13}} = (A^{(1/2)})_{11}\frac{a_{23}}{a_{13}} \geqslant 0.$$

(iii) $\delta(A) < 0$ and $\delta(A^S) < 0$. In this case $A^{(1/2)} = A^S$. $\delta(A^S) < 0$ implies $a^S_{11}a^S_{23} < a^S_{12}a^S_{13}$ and finally $(A^{(1/2)})_{12} = a^S_{12} > a^S_{11}a_{23}/a_{13} \geqslant 0$. □

Proposition 3.9. *The matrix $A^{(1)}$ defined in* (11) *is positive semidefinite. Furthermore, it is doubly non-negative if and only if* $\max\{\delta(A), \delta(A^S)\} \geqslant 0$.

Proof. The semidefinite positivity follows immediately by Proposition 3.8 and Lemma 3.3. Concerning the double non-negativity, also in this case it is enough to check the $(1, 2)$ entry. From (4), (5) and (11), we have

$$(A^{(1)})_{12} = (A^{(1/2)})_{23} - \frac{(A^{(1/2)})_{12}(A^{(1/2)})_{13}}{(A^{(1/2)})_{11}} = \frac{\delta(A^{(1/2)})}{(A^{(1/2)})_{11}}.$$

The assertion follows from (12). □

The following lemma is an elementary result, which we prove for the sake of completeness.

Lemma 3.10. *Let $C \in \mathscr{C}P^{(3)}$. Let* $\operatorname{rank} C = 2$ *and $\delta(C) \leqslant 0$. Then there exists a non-negative matrix V_C such that $C = V_C^{\mathrm{T}} V_C$, and*

$$V_C = \begin{bmatrix} v_{11} & v_{12} & v_{13}\\ v_{21} & v_{22} & 0\end{bmatrix}.$$

Proof. By assumption we have the inequalities $c_{11}c_{33} \geqslant c^2_{13}$, $c_{12}c_{13} \geqslant c_{11}c_{23}$. So we can easily obtain

$$c_{12}c_{33} \geqslant c_{13}c_{23}. \tag{13}$$

Now let us write $C = C_1 + C_2$ as follows:

$$C = \begin{bmatrix} \frac{(c_{13})^2}{c_{33}} & \frac{c_{13}c_{23}}{c_{33}} & c_{13}\\ \frac{c_{13}c_{23}}{c_{33}} & \frac{(c_{23})^2}{c_{33}} & c_{23}\\ c_{13} & c_{23} & c_{33}\end{bmatrix} + \begin{bmatrix} c_{11} - \frac{(c_{13})^2}{c_{33}} & c_{12} - \frac{c_{13}c_{23}}{c_{33}} & 0\\ c_{12} - \frac{c_{13}c_{23}}{c_{33}} & c_{22} - \frac{(c_{23})^2}{c_{33}} & 0\\ 0 & 0 & 0\end{bmatrix}.$$

C_1 is non-negative and has rank equal to 1, so we can write $C_1 = \underline{v}_1\underline{v}_1^{\mathrm{T}}$ for some $\underline{v}_1 \geqslant 0$. The same holds for C_2, since (13) implies

$$c_{12} - \frac{c_{13}c_{23}}{c_{33}} = \frac{c_{12}c_{33} - c_{13}c_{23}}{c_{33}} \geqslant 0.$$

So $C_2 = \underline{v}_2 \underline{v}_2^{\mathrm{T}}$ for some $\underline{v}_2 \geqslant 0$, where $(\underline{v}_2)_3 = 0$. Finally it suffices to define $V_C^{\mathrm{T}} = (\underline{v}_1 \;\; \underline{v}_2)$. □

Theorem 3.11. *Let A be a doubly non-negative matrix whose graph contains the dog-ear* $(1; 2, 3)$. *According to the notation in* (8) *and* (11), *if $\delta(A) \geqslant 0$, then A is completely positive if and only if $A^{(1)}$ is completely positive.*

Proof. Since $\delta(A) \geqslant 0$ we have $A = A^{(1/2)}$ and so $A^{(1)} = A^D(1)$. The sufficient condition is trivial, since the matrix A^L in (4) has rank 1, and so it is completely positive. By (5) we have the assertion.

Conversely there exists a non-negative matrix V such that $A = V^{\mathrm{T}}V$. Since $a_{1j} = 0$ for any $j \geqslant 4$, V can be chosen in the form

$$V = \begin{bmatrix} V' \\ V'' \end{bmatrix} = \begin{bmatrix} \underline{v}_1' & \underline{v}_2' & \underline{v}_3' & \mathbf{O}_{l,n-3} \\ \underline{0} & \underline{v}_2'' & \underline{v}_3'' & V_N'' \end{bmatrix}. \tag{14}$$

Now let $A' = V'^T V'$ and $A'' = V''^T V''$. A' and A'' are completely positive, and clearly

$$A = A' + A''. \tag{15}$$

Finally we define $H = W_H^{\mathrm{T}} W_H$, where $W_H = [I_l - \underline{v}_1' \underline{v}_1'^{+}]V'$. An easy check gives also $A^L = [\underline{v}_1' \underline{v}_1'^{+} V']^{\mathrm{T}} [\underline{v}_1' \underline{v}_1'^{+} V']$ and

$$A^L + H = A'. \tag{16}$$

Comparing (5), (15) and (16) we arrive at $A^D = A'' + H$. Since by Proposition 3.9 A^D is non-negative, applying Lemma 2.8 to the matrices A'' and H we can conclude rapidly. □

If $\delta(A) < 0$ we cannot apply Theorem 3.11. In this case, under the assumption of diminishability, at first we move from A to $A^{(1/2)}$.

Theorem 3.12. *Let A be a doubly non-negative matrix whose graph has the dog-ear* $(1; 2, 3)$. *If A is diminishable, then A is completely positive if and only if $A^{(1/2)}$ is completely positive.*

Proof. Since $A = A^{(1/2)} + \alpha(A)A^R$, the sufficient condition follows immediately by Lemma 2.8. Conversely, let A be completely positive. If $\delta(A) \geqslant 0$, it is $A = A^{(1/2)}$ and so there is nothing to prove. So let $\delta(A) < 0$. We can write $A = V^{\mathrm{T}}V$ as in (14). Thus

$$\operatorname{rank} A = \operatorname{rank} V \geqslant \operatorname{rank} (\underline{v}_1' \;\; \underline{v}_2' \;\; \underline{v}_3') + \operatorname{rank} V_N''.$$

Since A is diminishable, with regard to (2) and (3) we have

$$\operatorname{rank} A \leqslant \operatorname{rank} M + 1 \leqslant \operatorname{rank} N + 2,$$

furthermore $\operatorname{rank} V_N'' = \operatorname{rank} N$. If we write A as in (15), we conclude

$$\operatorname{rank} A' = \operatorname{rank} (\underline{v}_1' \;\; \underline{v}_2' \;\; \underline{v}_3') \leqslant 2.$$

It is not difficult to see that $(A)_{1i} = (A')_{1i}$ for any i, while $(A)_{23} \geqslant (A')_{23}$. This fact in particular implies $\delta(A') \leqslant \delta(A) < 0$. We can apply Lemma 3.10 to A'. This allows us to find another non-negative factorization of A (which will be called V again) in the form

$$V = \begin{bmatrix} v_{11} & v_{12} & 0 & \underline{0}^{\mathrm{T}} \\ v_{21} & v_{22} & v_{23} & \underline{0}^{\mathrm{T}} \\ \underline{0} & \underline{v}_2'' & \underline{v}_3'' & V_N'' \end{bmatrix}. \tag{17}$$

Let V^* be the matrix V without the first row, and let $A^* = V^{*T} V^*$. Clearly A^* is completely positive, and $A^* \leqslant A$. Furthermore $(A^*)^S = A^S$, and since, in general, for any positive semidefinite matrix C it is $C^S \leqslant C$, we get $A^S \leqslant A^* \leqslant A$. By virtue of Lemma 3.4 we can write

$$A^* = \beta A^S + (1-\beta)A \tag{18}$$

for some $\beta \in [0, 1]$. Now $\delta(A^*) \geqslant 0$. In fact

$$(A^*)_{11}(A^*)_{23} - (A^*)_{12}(A^*)_{13} = v_{21}^2(v_{22}v_{23} + \underline{v}_2''^T \underline{v}_3'') - (v_{21}v_{22})(v_{21}v_{23}) \geqslant 0.$$

If, for any $t \in [0, 1]$, we define

$$A(t) = tA^S + (1-t)A,$$

so that $A = A(0)$, $A^S = A(1)$, $A^* = A(\beta)$, $A^{(1/2)} = A(\alpha(A))$. It is only a check to see that

$$\delta(A(t)) = t\delta(A^S) + (1-t)\delta(A).$$

Thus $\delta(A(t))$ is monotone in t. The conditions $\delta(A) < 0$ and $\delta(A^*) \geqslant 0$ imply $\delta(A(t))$ increasing in t. This last fact implies $\delta(A^S) \geqslant 0$. By (12) we have $\delta(A^{(1/2)}) = 0$ and finally this implies $\alpha(A) \leqslant \beta$. Finally (18) gives

$$A^{(1/2)} = A^* + (\beta - \alpha(A))A^R.$$

Proposition 3.8 and Lemma 2.8 give the assertion. □

Remark 3.13. In the proof of Theorem 3.12 we have seen that the complete positivity of A and the condition $\delta(A) < 0$ imply $\delta(A^S) \geqslant 0$. So in particular, given a doubly non-negative diminishable matrix A, if $\delta(A)$ and $\delta(A^S)$ are both negative, we can conclude that A is not completely positive. The same fact is no longer true if we remove the diminishability assumption; in particular for non-singular matrices, as we will see in Section 5.

Combining together Theorems 3.11 and 3.12, and Remark 3.13 we obtain the following.

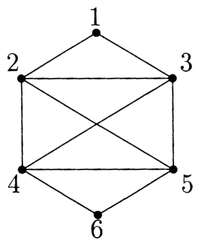
Corollary 3.14. *Let $A \in \mathscr{DN}^{(n)}$ such that $G(A)$ has the dog-ear* $(1; 2, 3)$. *According to* (11), *if A is diminishable, A is completely positive if and only if $A^{(1)}$ is completely positive.*

Proof. If $\delta(A) \geqslant 0$, the assertion follows from Theorem 3.11. If $\delta(A) < 0$ and $\delta(A^S) \geqslant 0$, by virtue of Theorem 3.12, $A \in \mathscr{CP}^{(n)}$ if and only if $A^{(1/2)} \in \mathscr{CP}^{(n)}$. Now (12) gives $\delta(A^{(1/2)}) = 0$. So we finish by applying Theorem 3.11 to $A^{(1/2)}$, since $[A^{(1/2)}]^{(1)} = A^{(1)}$. Finally if $\delta(A)$ and $\delta(A^S)$ are both negative, $A \notin \mathscr{CP}^{(n)}$ (see Remark 3.13), but $A^{(1)} \notin \mathscr{CP}^{(n)}$ too, since Proposition 3.9 claims that $A^{(1)}$ is not even doubly non-negative. $\square$

Example 3.15. Let us consider the following matrices:

$$A = \begin{bmatrix} 3 & 3 & 3 & 0 & 0 & 0 \\ 3 & 11 & 1 & 4 & 1 & 0 \\ 3 & 1 & 7 & 2 & 1 & 0 \\ 0 & 4 & 2 & 6 & 4 & 2 \\ 0 & 1 & 1 & 4 & 5 & 4 \\ 0 & 0 & 0 & 2 & 4 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 3 & 3 & 0 & 0 & 0 \\ 3 & 17/2 & 1 & 4 & 1 & 0 \\ 3 & 1 & 7 & 1 & 1 & 0 \\ 0 & 4 & 1 & 6 & 4 & 2 \\ 0 & 1 & 1 & 4 & 5 & 4 \\ 0 & 0 & 0 & 2 & 4 & 4 \end{bmatrix}$$

whose common graph has the dog-ears (1; 2, 3) and (6; 4, 5).



Both matrices are doubly non-negative and diminishable. Moreover $\delta(A) = \delta(B) = -6 < 0$. Using (9)–(11) we can compute

$$A^{(1)} = \begin{bmatrix} 6 & 0 & 4 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 \\ 4 & 2 & 6 & 4 & 2 \\ 1 & 1 & 4 & 5 & 4 \\ 0 & 0 & 2 & 4 & 4 \end{bmatrix}, \qquad B^{(1)} = \begin{bmatrix} 5 & -1 & 4 & 1 & 0 \\ -1 & 2 & 1 & 1 & 0 \\ 4 & 1 & 6 & 4 & 2 \\ 1 & 1 & 4 & 5 & 4 \\ 0 & 0 & 2 & 4 & 4 \end{bmatrix}.$$

$A^{(1)}$ is doubly non-negative and its graph is completely positive. Therefore $A^{(1)}$ is completely positive, and so is A, by virtue of Corollary 3.14. On the other hand, $B^{(1)}$ (and B) is not completely positive.

4. Non-crossing cycles

If the graph of the diminished matrix $A^{(1)}$ still contains a dog-ear, the natural idea is to apply again the deletion of the dog-ear, in order to obtain a still smaller matrix. Unfortunately, in general, the diminishability of A does not imply the diminishability of $A^{(1)}$. For example the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 & 0 & 0 \\ 1 & 2 & 3 & 1 & 1 & 0 \\ 2 & 3 & 5 & 1 & 1 & 0 \\ 0 & 1 & 1 & 5 & 1 & 2 \\ 0 & 1 & 1 & 1 & 6 & 2 \\ 0 & 0 & 0 & 2 & 2 & 2 \end{bmatrix}$$

whose graph has two dog-ears, is diminishable. The diminished matrix is

$$A^{(1)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 5 & 1 & 2 \\ 1 & 1 & 1 & 6 & 2 \\ 0 & 0 & 2 & 2 & 2 \end{bmatrix}$$

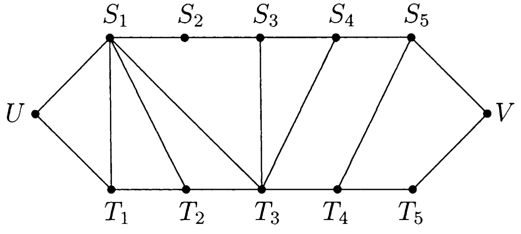
whose graph has the dog-ear $(5; 3, 4)$. Re-ordering rows and columns we can put it in the form (2). But the re-ordered matrix is not diminishable. This undesired fact can happen when two dog-ears are "too far" from each other, while the same does not happen if the dog-ears are "consecutive", namely if $(1; 2, 3)$ is a dog-ear for $G(A)$, then $(2; 3, 4)$ is a dog-ear for $G(A^{(1)})$.

In particular, the graph we are going to define is formed by a sequence of consecutive dog-ears.

Definition 4.1. Let G be a graph with n vertices. We say G to be a *non-crossing cycle* if the set $\mathscr{W}$ of its vertices can be shared into four subsets $\mathscr{U} = \{U\}$, $\mathscr{S} = \{S_1, \dots, S_s\}$, $\mathscr{T} = \{T_1, \dots, T_t\}$, $\mathscr{V} = \{V\}$, such that

(i) U is connected only to S_1 and T_1;
(ii) V is connected only to S_s and T_t;
(iii) for any $i \in \langle s-1 \rangle$, S_i is connected to S_{i+1} and there are no other connections between S_i's;
(iv) for any $j \in \langle t-1 \rangle$, T_j is connected to T_{j+1} and there are no other connections between T_j's;
(v) if S_i and S_h ($i \leqslant h$) are connected, respectively, to T_j and T_k, then $j \leqslant k$. In other words there are no crossings between edges connecting $\mathscr{S}$ and $\mathscr{T}$.

An example of non-crossing cycle is as follows.



It will be useful also to define a lightly larger class of graphs, called *generalized non-crossing cycles*, in which the property (i) is substituted by

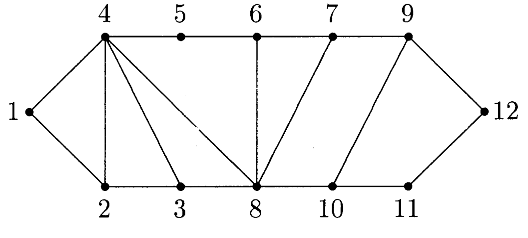
(i′) U is connected only to at least one of S_1 and T_1.

Clearly, a generalized non-crossing cycle can be easily reduced to a non-crossing cycle by deletion of some leaves.

In order to number from 1 to n the vertices of a generalized non-crossing cycle, let us at first consider two arbitrary vertices $S_i \in \mathscr{S}$ and $T_j \in \mathscr{T}$. By assumption (v), exactly one of the following cases holds:

$$\begin{aligned}&\text{(a) } S_i \text{ is connected to some of } T_k\text{'s for } k > j;\\ &\text{(b) } T_j \text{ is connected to some of } S_h\text{'s for } h > i;\\ &\text{(c) neither (a) nor (b).}\end{aligned} \tag{19}$$

Now we define the assignment $\nu : \mathscr{W} \longrightarrow \langle n\rangle$ as follows: let $\nu(U) = 1$, $\nu(V) = n$. Then we consider the vertices S_1 and T_1, and we look at (19) with $i = j = 1$. If (a) holds, we assign $\nu(T_1) = 2$, while if (b) or (c) holds, we assign $\nu(S_1) = 2$. Then we go on: if it were $\nu(T_1) = 2$, we have to consider the vertices S_1 and T_2, repeat the check (19) and again if (a) holds we define $\nu(T_2) = 3$, otherwise $\nu(S_1) = 3$. The general rule is to compare the vertices S_i and T_j and assign $\nu(T_j) = i + j$ if (a) holds, while $\nu(S_i) = i + j$ otherwise. For example, the previous non-crossing cycle would be numbered as follows.



Actually when we finish numbering the set $\mathscr{S}$, there are still some T_k's to be numbered (T_4 and T_5 in the example). In this case the rule is to complete the numbering respecting their natural order (so $\nu(T_4) = 10$, $\nu(T_5) = 11$). The vertices of a generalized non-crossing cycle will be indifferently identified by their "name" (i.e. U, V, $S_1, \ldots$) or their associated number $(1, \ldots, n)$. This numbering is particularly useful since, after the deletion of the first dog-ear, the vertex 2 becomes the vertex of a new dog-ear or a leaf. The same holds for the following vertices, after a suitable number of deletions. In general, when dealing with matrices with a generalized non-crossing cycle as associated graph, we always assume the vertices to be numbered as said.

However, there is a cost for this numbering. In fact the first dog-ear will be $(1; 2, l)$, where l can be in general different from 3. In the previous example it was $(1; 2, 4)$. This means we have to change a bit the form (2) and the matrices A^L and A^D defined in (4) and (5), by writing l instead of 3. The same is necessary for (8), while no changes are due for (6), (7), (10) and (11). Also the notion of diminishability need not to be modified. However in this case it becomes quite simpler, and, in a sense, superfluous, by virtue of the following result.

Theorem 4.2. *Let $A \in \mathscr{DN}^{(n)}$ such that $G(A)$ is a non-crossing cycle. Then A is diminishable if and only if it is singular.*

Proof. Clearly we just have to prove the sufficient conditions. Actually, according to (3) we just prove M to be non-singular. So let $(1; 2, l)$ be the dog-ear whose vertex is 1. And let us define the map $\sigma : \{1, \ldots, n-2\} \longrightarrow \{3, \ldots, n\}$ as follows:

$$\begin{aligned}
&\sigma(1) = l,\\
&\sigma(i) = \nu(S_{h+1}) \quad \text{if } i = \nu(S_h), \quad h < s = |\mathscr{S}|,\\
&\sigma(i) = \nu(T_{k+1}) \quad \text{if } i = \nu(T_k), \quad k < t = |\mathscr{T}|,\\
&\sigma(i) = n \quad \text{if } i = \nu(S_s).
\end{aligned}$$

Note that $i = \nu(T_t)$ is impossible, since $\nu(T_t)$ is always equal to $n-1$. (In the previous example it would be

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 3 & 8 & 5 & 6 & 7 & 9 & 10 & 12 & 11 \end{bmatrix},$$

where we have borrowed the notation of permutations.)

Since the graph is a non-crossing cycle, for any $p < q \in \langle n-2 \rangle$, the vertices p and $\sigma(p)$ are connected, while p and $\sigma(q)$ are not connected. This is quite evident if $p = 1$ or if p and q are both in $\mathscr{S}$ or both in $\mathscr{T}$, while if $p \in \mathscr{S}$ and $q \in \mathscr{T}$, since $p < q$, the numbering introduced using (19) assures the vertex p not to be connected to any vertex following q. Similarly if $p \in \mathscr{T}$ and $q \in \mathscr{S}$.

Now let us consider the matrix A^σ obtained by re-arranging the columns of A in this way:

$$(A)^\sigma_{*,1} = (A)_{*,1}, \quad (A)^\sigma_{*,2} = (A)_{*,2}, \quad (A)^\sigma_{*,i+2} = (A)_{*,\sigma(i)} \ \ (i \leqslant n-2).$$

Finally let us consider the square submatrix $A^\sigma(\{n-1, n\}, \{1, 2\})$. Its diagonal entries are

$$(A^\sigma)_{i,i+2} = (A)_{i,\sigma(i)} \neq 0 \quad \forall i \in \langle n-2 \rangle$$

since, as seen, in $G(A)$ i is connected to $\sigma(i)$. On the other hand, all the upper-diagonal entries are 0. In fact if $i < j$, it is $(A^\sigma)_{i,j+2} = (A)_{i,\sigma(j)} = 0$ since i is not connected to $\sigma(j)$. In particular $A^\sigma(\{n-1, n\}, \{1, 2\})$ is lower triangular and non-singular. This means that the matrix $\begin{bmatrix} S^{\mathrm{T}} \\ M \end{bmatrix}$ in (3) has rank equal to $n-2$. The semidefinite positivity of A implies $S \in R(M)$, that means M is non-singular. $\square$

Remark 4.3. If $G(A)$ is only a generalized non-crossing cycle, Theorem 4.2 does not hold any longer, i.e. there exist singular matrices with a leaf which are not diminishable. However this do not will be a problem in the following argument, since the deletion of a leaf does not require diminishability.

Now let A be singular and $G(A)$ a generalized non-crossing cycle. If the vertex 1 is a leaf, we can define the diminished matrix $A^{(1)}$ as in (1). Similarly if the vertex 1 is the vertex of a dog-ear, we define $A^{(1)}$ as in (11). In both cases one can prove that $A^{(1)}$ is singular and $G(A^{(1)})$ is again a generalized non-crossing cycle. By virtue of either Theorem 2.7 or Corollary 3.14, in order to investigate complete positivity of A we can reduce the problem to $A^{(1)}$. Now there are two possibilities: either $A^{(1)}$ is not non-negative, and so it is not completely positive and we have finished, or it is non-negative. In this last case we define

$$A^{(2)} = (A^{(1)})^{(1)}$$

and we can apply again Theorem 2.7 or Corollary 3.14 to the matrix $A^{(1)}$ and reduce the problem to $A^{(2)}$. Now we can go on deleting leaves and/or dog-ears, defining inductively $A^{(j)} = (A^{(j-1)})^{(1)}$. If at a certain point we meet with a matrix $A^{(j)}$ which fails to be non-negative, we have to stop and conclude A not to be completely positive. On the other hand, if we can arrive till $A^{(n-4)}$ and this last matrix comes out non-negative, since its order is 4, it is completely positive and so is A. So we have:

Corollary 4.4. *Let A be an $n \times n$ doubly non-negative singular matrix with a generalized non-crossing cycle as associated graph. Then A is completely positive if and only if, for any $j \leqslant n-4$, $A^{(j)}$ is non-negative.*

A particular case of non-crossing cycle is the graph of a pentadiagonal matrix all of whose second-super-diagonal entries are non-zero.

Example 4.5. Let us consider the matrix

$$A = \begin{bmatrix} 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 13 & 3 & 2 & 0 & 0 \\ 0 & 1 & 3 & 9/2 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 5 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 \end{bmatrix}.$$

A is doubly non-negative and singular. We find

$$A^{(1)} = \begin{bmatrix} 4/5 & 0 & 1 & 0 & 0 & 0 \\ 0 & 6 & 3 & 2 & 0 & 0 \\ 1 & 3 & 9/2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 5 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 2 & 2 & 2 \end{bmatrix},$$

$$A^{(2)} = \begin{bmatrix} 6 & 3 & 2 & 0 & 0 \\ 3 & 13/4 & 1 & 1 & 0 \\ 2 & 1 & 5 & 1 & 2 \\ 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 2 & 2 & 2 \end{bmatrix}, \qquad A^{(3)} = \begin{bmatrix} 7/4 & 0 & 1 & 0 \\ 0 & 13/3 & 1 & 2 \\ 1 & 1 & 3 & 2 \\ 0 & 2 & 2 & 2 \end{bmatrix}.$$

Since $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$ are non-negative, by virtue of Corollary 4.4 A is completely positive.

Remark 4.6. For completely positive singular matrices whose graph is a non-crossing cycle, it is even possible to furnish a non-negative factorization. In fact if $\tilde{A}^{(i)}$ denotes the $n \times n$ matrix obtained from $A^{(i)}$ by pre-bordering it with a suitable number of zero rows and columns, we can write

$$A = [A - \tilde{A}^{(1)}] + [\tilde{A}^{(1)} - \tilde{A}^{(2)}] + \cdots + [\tilde{A}^{(n-5)} - \tilde{A}^{(n-4)}] + \tilde{A}^{(n-4)}.$$

Actually, each summand at the right-hand side has rank less than or equal to 2 and is completely positive. By bringing together the factorizations of all these summands, we have a factorization for A. With regard to Example 4.5 we have

$$V^{\mathrm{T}} = \begin{bmatrix} \sqrt{\frac{10}{7}} & \sqrt{\frac{4}{7}} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\frac{72}{35}} & \sqrt{\frac{1}{7}} & \sqrt{\frac{4}{5}} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{7} & 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{5}{4}} & \sqrt{\frac{3}{2}} & \sqrt{\frac{7}{4}} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{13}{3}} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{4}{7}} & \sqrt{\frac{3}{13}} & \sqrt{\frac{200}{91}} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{12}{13}} & \sqrt{\frac{14}{13}} \end{bmatrix}.$$

5. Non-diminishable matrices

Looking at Corollary 3.14, a question arises spontaneously: is it possible to improve in some way the result by removing the diminishability assumption? At the moment we do not know a complete answer. In this section we want to explain where is the critical point which prevents this expected improvement. We just recall that Theorem 3.11 does not require any diminishability. So the problem arises only when $\delta(A) < 0$. Let us consider the matrix

$$A = \begin{bmatrix} 7 & 4 & 2 & 0 & 0 \\ 4 & 5 & 1 & 1 & 0 \\ 2 & 1 & 5 & 1 & 2 \\ 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 2 & 2 & 2 \end{bmatrix}.$$

Actually $\delta(A)$ and $\delta(A^S)$ are both negative. Nevertheless A is completely positive since $A = V^{\mathrm{T}}V$ where

$$V^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & \sqrt{6} & 0 & 0 & 0 \\ 1 & \sqrt{\frac{3}{4}} & \sqrt{\frac{3}{2}} & \sqrt{\frac{7}{4}} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{13}{3}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{4}{7}} & \sqrt{\frac{3}{13}} & \sqrt{\frac{200}{91}} \\ 0 & 0 & 0 & 0 & \sqrt{\frac{12}{13}} & \sqrt{\frac{14}{13}} \end{bmatrix}.$$

Clearly, the assertion of Remark 3.13 does not hold because A is non-singular and so non-diminishable. In a similar way it is not difficult to find a completely positive (non-singular) matrix in the form (2) such that $\delta(A) < 0$, $\delta(A^S) \geqslant 0$ but the matrix $A^{(1/2)}$ defined in (10) fails to be completely positive. So also Theorem 3.12 does not hold for non-singular matrices. What is the reason for this? Actually the natural extension of Theorem 3.12 to the non-diminishable case is the following.

Proposition 5.1. *Let A be an $n \times n$ doubly non-negative matrix whose graph contains the dog-ear* $(1; 2, 3)$. *With regard to* (3), (6) *and* (8), *A is completely positive if and only if there exists a completely positive matrix A^* such that $A^S \leqslant A^* \leqslant A$ and $\delta(A^*) \geqslant 0$.*

Proof. The only non-trivial part is the necessary condition when $\delta(A) < 0$. The proof is quite similar to the necessary condition in the proof of Theorem 3.12. We just have some problems to arrive at (17). In fact if we write A as in (15), now it can be rank $A' = 3$. In this case we can find $\lambda > 0$ so that $A' - \lambda E_{33}$ has rank 2. So we can rewrite (15) with $A' - \lambda E_{33}$ instead of A' and, clearly, $A'' + \lambda E_{33}$ instead of A''. The matrix A^* defined just after (17) has the required properties. $\square$

So if we define the closed set

$$\mathscr{M} = \{A^* \in \mathscr{DN}^{(n)} \mid A^S \leqslant A^* \leqslant A,\ \delta(A^*) \geqslant 0\}$$

and $\overline{\mathscr{M}}$ denotes the set of the maximal elements of $\mathscr{M}$, by virtue of Lemma 2.8, Proposition 5.1 can be changed to:

Corollary 5.2. *Let $A \in \mathscr{DN}^{(n)}$ and let $G(A)$ contain the dog-ear* $(1; 2, 3)$. *Then $A \in \mathscr{CP}^{(n)}$ if and only if $\overline{\mathscr{M}} \cap \mathscr{CP}^{(n)} \neq \emptyset$.*

In general the condition $\overline{\mathscr{M}} \cap \mathscr{CP}^{(n)} \neq \emptyset$ can be checked only when $\overline{\mathscr{M}}$ is a finite set. And this is the crucial point: if $\mathscr{M} \neq \emptyset$ and A is diminishable, $\mathscr{M}$ is a one-dimensional set, so $\overline{\mathscr{M}}$ has a unique element, that is exactly the matrix $A^{(1/2)}$ defined in (10). So we get again Theorem 3.12. On the other hand, if A is not diminishable, in general $\mathscr{M}$ is a three-dimensional set and so $\overline{\mathscr{M}}$ can be an infinite set.

In this case, the idea would be to find a way to define a matrix $A^* \in \overline{\mathscr{M}}$ so that $\{A^*\} \cap \mathscr{CP}^{(n)} \neq \emptyset$ if and only if $\overline{\mathscr{M}} \cap \mathscr{CP}^{(n)} \neq \emptyset$. This would allow us to find a result quite similar to Theorem 3.12 also for the non-diminishable case.

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